A Steady-State Dynamics, Direct Procedure for a Bar with Bloch-Floquet Periodicity

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Overview

The goal is to develop a numerical tool that can be used to predict the direct, steady-state dynamics response of an infinite 1D bar structure subject to an oblique incident wave using Bloch-Floquet periodicity. Bloch-Floquet theory is used to extend the results of a *unit cell* into the infinite domain by imposing periodic boundary conditions (PBCs) on the edges of the bar. Bloch-Floquet theory is required when a structure is loaded obliquely as shown in Figure 1 below which generates traveling waves across the structure and care must be taken to ensure the phase of the wave is incorporated in the constraint. Typical applications of Bloch-Floquet theory are the design of band-gap structures [2] and surface acoustic wave (SAW) structures [3].

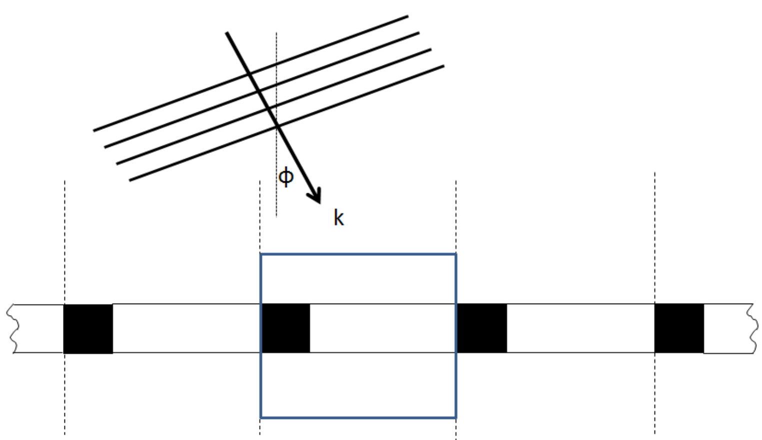

Figure 1. Infinitely periodic bar model of two different material properties subject to an impinging wavevector, *k*

The resulting loading causes a traveling wave to develop in the structure. The wavenumber of the loading is a property of the acoustic medium,

$$k = \frac{\omega}{c}$$

where ω is the circular frequency of excitation and *c* is the wave speed of the acoustic medium. The trace wavelength along the length of the bar is

$$k_x = k \sin\phi$$

In lieu of the acoustic medium in the analysis, this can be directly applied as a force at the nodes of the finite elements in the model. The applied force becomes

$$F = F_0 e^{ik_x x} = F_0\{\cos k_x x + i \sin k_x x\}$$

where *x* is the coordinate of the bar. This in-plane loading can be conceptualized as in Figure 2.

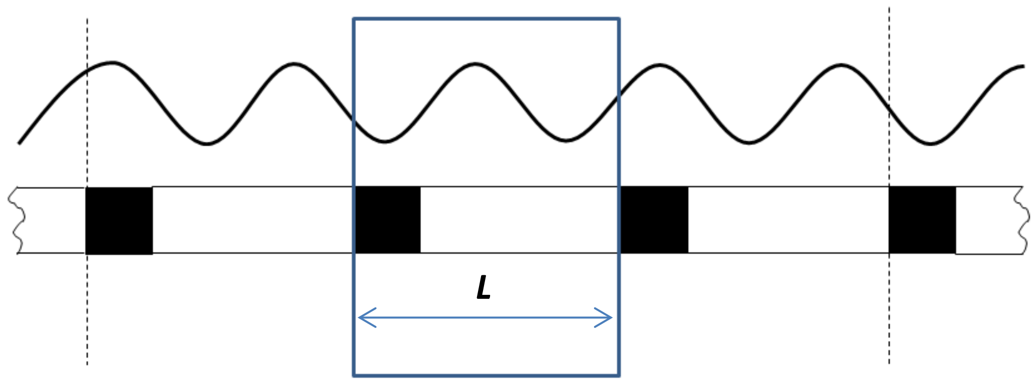

Figure 2. Infinitely periodic bar model of two different material properties subject to an impinging trace wave

The key is than an infinite domain can be represented by a single unit cell, as shown by a rectangle in Figures 1 and 2. The unit cell is the only domain modeled by finite elements. The results of the other cells can be predicted using Bloch-Floquet theory. Here, the theory is used to impose the following constraint,

$$\boldsymbol{u}(L) = \boldsymbol{u}(0)e^{-ik_xL}$$

which is used to ensure a proper phase relation between over the length of the unit cell. The magnitude will always be equal, however. In order to impose this constraint, however, the trace wavenumber in the structure must be known *a priori*.

Derivation of the weak form of the linear momentum balance for a steady-state dynamics direct procedure

The direct approach seeks to solve the linear momentum balance with an acceleration term directly. The steady-state assumption is made during the Galerkin formulation.

The strong form of the linear momentum balance is defined as [1]

$$\nabla y \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a},$$

or alternatively in index notation,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i.$$

By the principle of virtual work (PVW), an arbitrary test function, δu_i , is constructed over the volume

$$\int_V \left\{ \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i - \rho a_i \right\} \delta u_i \, dV = 0$$

Neglecting body forces and separating terms gives

$$\int_V \delta u_i \frac{\partial \sigma_{ij}}{\partial x_j} dV - \int_V \delta u_i \, \rho \ddot{u}_i \, dV = 0$$

Integrating by parts on the first terms and applying the divergence theorem gives

$$\int_S n_j \sigma_{ji} \delta u_i \, dS - \int_V \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} dV - \int_V \delta u_i \, \rho \ddot{u}_i dV = 0$$

Applying the traction BC gives

$$\int_S t_i^* \delta u_i \, dS - \int_V \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} dV - \int_V \delta u_i \, \rho \ddot{u}_i dV = 0$$

Reordering gives us the weak form

$$\int_V \delta u_i \, \rho \ddot{u}_i dV + \int_V \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} dV - \int_S t_i^* \delta u_i \, dS = 0$$

Discretizing with a Galerkin scheme of

$$u_k = N^a u_k^a$$

$$\delta u_k = N^a \delta u_k^a$$

$$\frac{\partial \delta u_i}{\partial x_j} = \frac{\partial N^a}{\partial x_j} \delta u_i^a$$

And simplifying gives

$$\delta u_i^b \left\{ \ddot{u}_i^a \int_V \rho N^b \, N^a dV + u_i^a \int_V C_{ijkl} \frac{\partial N^b}{\partial x_j} \frac{\partial N^a}{\partial x_j} dV - \int_S N^b t_i^* \, dS \right\} = 0$$

which must be true for any arbitrary δu_i^b. This can be written in matrix form

$$\boldsymbol{M}^{ba} \ddot{\boldsymbol{u}}^a + \{ i \boldsymbol{C}_s^{ba} + \boldsymbol{K}^{ba} \} \boldsymbol{u}^a = \boldsymbol{F}^b \tag{1}$$

Where a structural damping term has been introduced. The matrices are defined as

$$\boldsymbol{M}^{ba} = \int_V \rho N^b \, N^a dV$$

$$\boldsymbol{K}^{ba} = \int_V C_{ijkl} \frac{\partial N^b}{\partial x_j} \frac{\partial N^a}{\partial x_j} dV$$

$$\boldsymbol{C}_s^{ba} = \int_V s \, C_{ijkl} \frac{\partial N^b}{\partial x_j} \frac{\partial N^a}{\partial x_j} dV$$

$$\boldsymbol{F}^b = \int_S N^b t_i^* \, dS$$

Assuming simple harmonic motion in the case of steady state dynamics, we introduce harmonic displacements and forces

$$\Delta \boldsymbol{u}^a = \{ \mathrm{Re}[\boldsymbol{u}^a] + i \, \mathrm{Im}[\boldsymbol{u}^a] \} e^{i\omega t}$$

$$\Delta \boldsymbol{F}^b = \{ \mathrm{Re}[\boldsymbol{F}^b] + i \, \mathrm{Im}[\boldsymbol{F}^b] \} e^{i\omega t}$$

where ω is the circular frequency of excitation. Also note that the acceleration becomes

$$\Delta \ddot{\boldsymbol{u}}^a = -\omega^2 \{ \mathrm{Re}[\boldsymbol{u}^a] + i \, \mathrm{Im}[\boldsymbol{u}^a] \} e^{i\omega t}$$

This can be substituted into Eq. (1) to give

$$\begin{aligned} \boldsymbol{M}^{ba} \left(-\omega^2 \{ \mathrm{Re}[\boldsymbol{u}^a] + i \, \mathrm{Im}[\boldsymbol{u}^a] \} e^{i\omega t} \right) + \{ i \boldsymbol{C}_s^{ba} + \boldsymbol{K}^{ba} \} \left(\{ \mathrm{Re}[\boldsymbol{u}^a] + i \, \mathrm{Im}[\boldsymbol{u}^a] \} e^{i\omega t} \right) \\ = \left(\{ \mathrm{Re}[\boldsymbol{F}^b] + i \, \mathrm{Im}[\boldsymbol{F}^b] \} e^{i\omega t} \right) \end{aligned}$$

The $e^{-i\omega t}$ term drops out, and this expression can be split into real and complex expressions for numerical efficiency. However for the simple implementation at hand, we will solve the complex system of equations in MATLAB directly. The complex system is

$$\boldsymbol{M}^{ba}\left(-\omega^2 \boldsymbol{u}^a e^{i\omega t}\right) + \{i\boldsymbol{C}_s^{ba} + \boldsymbol{K}^{ba}\}\left(\boldsymbol{u}^a e^{i\omega t}\right) = \left(\boldsymbol{F}^b e^{i\omega t}\right)$$

Simplifying gives

$$\{\boldsymbol{K}^{ba} + i\boldsymbol{C}_s^{ba} - \omega^2 \boldsymbol{M}^{ba}\}\, \boldsymbol{u}^a = \boldsymbol{F}^b \tag{2}$$

Now Eq. (2) is solved in MATLAB for displacements where the RHS is known for each node, and the LHS is computed for each frequency of interest in the analysis.

Imposing of Bloch-Floquet Periodicity

The Bloch-Floquet constraint is imposed with a master-slave formulation in MATLAB. With this approach, the node at the beginning of the unit cell (i.e. x=0) is used as the master, and the node at x=L is the slave in the constraint.

$$\boldsymbol{u}(L) = \boldsymbol{u}(0)e^{-ik_xL}$$

A transformation matrix is constructed [4] to relate the original unknowns u_1 … u_L to the new set in which u_L is omitted.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{L-1} \\ u_L \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ e^{-ik_xL} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{L-1} \end{bmatrix}$$

In compact form,

$$\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}} \tag{3}$$

Replacing Eq. (3) into Eq. (2) and pre-multiplying by $\mathbf{T}^T$ gives the modified system

$$\widehat{\boldsymbol{A}}\,\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{F}} \tag{4}$$

Where

$$\widehat{\boldsymbol{A}} = \boldsymbol{T}^T\,\{\boldsymbol{K}^{ba} + i\boldsymbol{C}_s^{ba} - \omega^2 \boldsymbol{M}^{ba}\}\,\boldsymbol{T}$$

$$\widehat{\boldsymbol{F}} = \boldsymbol{T}^T\,\boldsymbol{F}^b$$

This modified system in Eq. (4) is solved and the displacements are recovered by Eq. (3). This method is computationally intensive for large problems, however, because of large bandwidth of the system matrix.

Implementation into MATLAB

A template was downloaded from solidmechanics.org for a bar model and expanded to:

- Solve the steady state dynamic procedure directly as shown in Eq. (2)
- Impose Bloch-Floquet boundary conditions per Eq. (4)
- Include multiple bar properties in the unit cell

- Ability to model any number of unit cells for validation purposes
- Ability to impose oblique acoustic loading

Validation of Implementation

Validation is performed with an analytical model which uses a modal approach to solve the problem. The solution was derived by a fellow colleague and the FEA model is used as a sanity-check. The modal approach uses 2001 terms and the FEA mesh uses 720 two node linear finite elements per unit cell. The following results are shown at 1 kHz with 8 sections and the error in the magnitude of the response is less than 0.1 dB. Using more terms in the modal solution and increasing the resolution of the mesh will reduce this error further.

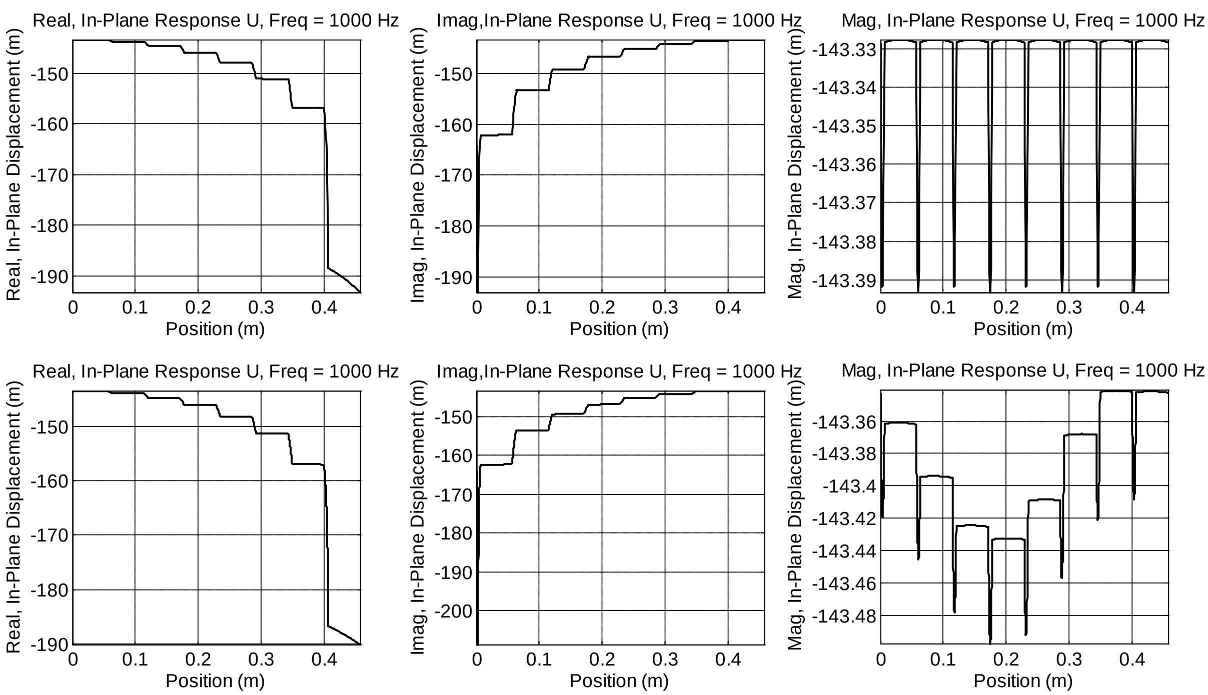

Figure 3. Comparison of analytic to FEA solution at 1 kHz. Top: analytic, Bottom: FEA.

Alternatively, the results at 5 kHz are shown below and no noticeable error is present.

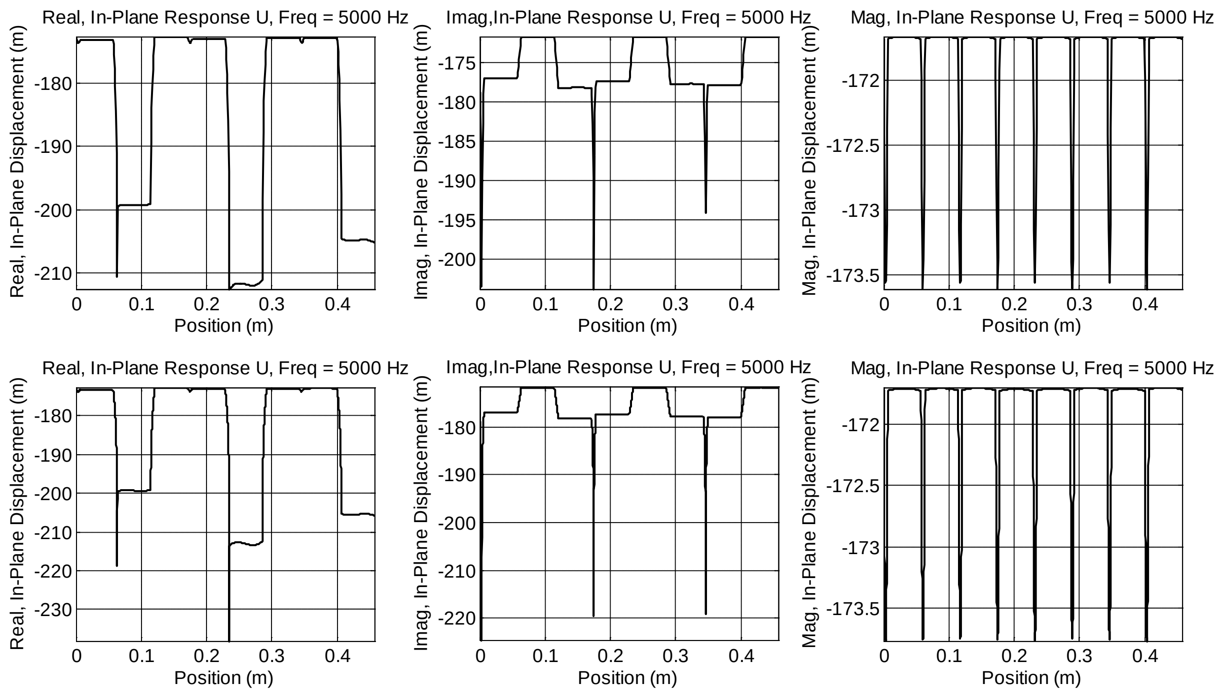

Figure 4. Comparison of analytic to FEA solution at 5 kHz. Top: analytic, Bottom: FEA.

Further, results at 500 Hz, which are off by about 0.05 dB.

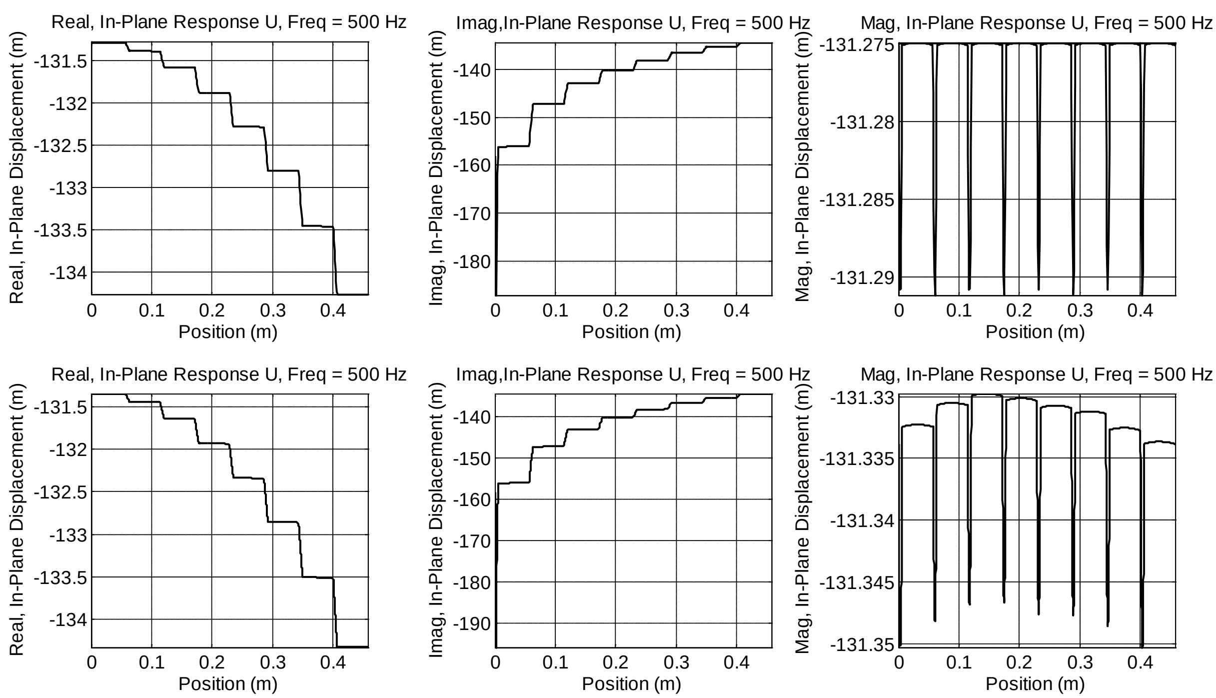

Figure 5. Comparison of analytic to FEA solution at 500 Hz. Top: analytic, Bottom: FEA.

Wavenumber-Frequency Diagrams

With the FEA code validated, it is worthwhile to sweep through a range of frequencies and generate a wavenumber-frequency diagram which will show the traveling trace wave and also the periodically shifted wave.

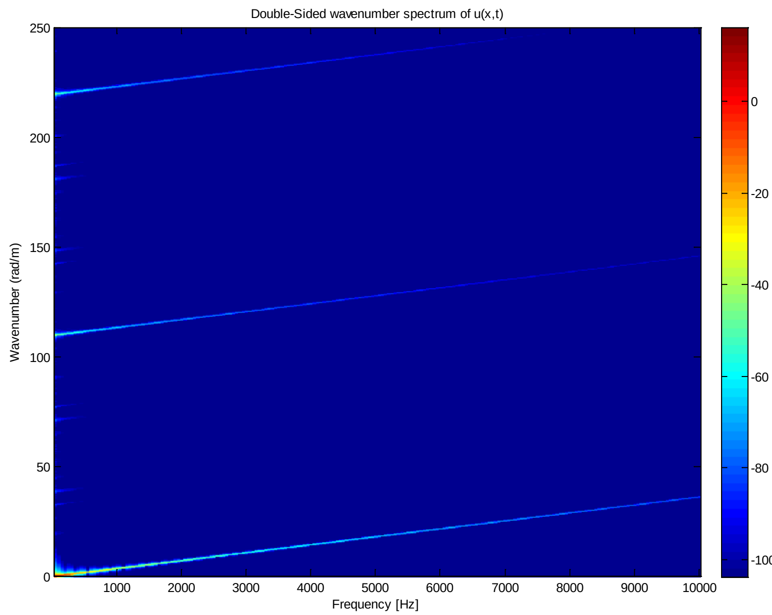

Figure 6. Wavenumber-Frequency diagram for the two-section bar model.

Figure (6) shows the vertically shifted wave which is defined by the unit cell length, $k_{shift} = \frac{2\pi}{L}$. Also note that the inverse slope of the visible lines is equal to the wave speed of the trace wave, in this case it is 301 m/s.

Conclusion

A direct solution to a steady-state dynamics implementation of a bar element was developed subject to Bloch-Floquet periodicity. Results were compared with an analytical model and less than a 0.1 dB difference in complex magnitude was found in all cases. This tool can be used to help design simple periodic structures and quickly perform parametric studies, such as to generate wavenumber-frequency diagrams as shown in the previous section.

References

[1] Bower, A.F. *Applied Mechanics of Solids*, 2010

[2] Diaz, A.R., Haddow, A.G., Ma, L. *Design of Band-Gap Grid Structures*, Structural Multidisc Optimization (2005) **29**:418-431

[3] Hofer, M.,et. al. *Finite Element Simulation of Wave Propagation in Periodic Piezoelectric SAW Structures*, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, 2000

[4] Felippa, Carlos, *Introduction to Finite Element Methods*, 2004.